QP Code: 24/PT/13/VIIIB

POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-8B : GRAPH THEORY

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Define the distance between two vertices of a simple graph.

   b) Define a directed graph with example.

   c) If G is a forest with n vertices and k components, find the number of edges of G.

   d) Prove that if a graph contains exactly two vertices u and v of odd degree, then there exists a path between u and v.

   e) Show that every connected graph has at least one spanning tree.

   f) Define adjacency matrix with an example.

   g) Define leaf and internal vertex of a rooted tree.

2. a) State and prove the necessary and sufficient condition of a connected graph to be Eulerian. 1 + 5

   b) Define Hamiltonian path. Does the Peterson graph have a Hamiltonian path ? Justify your answer. 1 + 3

3. a) A cycle on n vertices is isomorphic to its complement. Find the value of n. 4

   b) Prove that in any digraph G, the sum of the in-degrees of the vertices = the sum of the out-degrees of the vertices = the number of arcs in G. 6

4. a) Can a simple graph have the degree sequence (2, 2, 4, 4, 4) ? Justify your answer. 3

   b) Prove that a simple graph is bipartite if and only if it has no odd cycle. 7

5. a) Prove that if T is a binary tree of height h having k leaves, then $k \le 2^h$. 6

   b) Let T be a tree with even number of edges. Prove that T must contain at least one vertex of even degree. 4

6. a) State and prove Euler's formula for simple connected planar graphs. 5

   b) For any simple connected planar graph with n vertices and e edges, prove that $e \le 3n - 6$. 5

7. Let A be the adjacency matrix of a simple graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$. Also let $Q = D + A$, where $D = diag(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with d_i being the degree of vertex v_i in G, $1 \le i \le n$.

   a) Prove that $Q = MM^T$, where M is the incidence matrix of G and M^T is the transpose of M. 4

   b) Prove that Q is positive semi-definite. 6